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Parameter Estimation of Linear Sensorimotor Synchronization Models: Phase Correction, Period Correction, and Ensemble Synchronization

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Abstract

Linear models have been used in several contexts to study the mechanisms that underpin sensorimotor synchronization. Given that their parameters are often linked to psychological processes such as phase correction and period correction, the fit of the parameters to experimental data is an important practical question. We present a unified method for parameter estimation of linear sensorimotor synchronization models that extends available techniques and enhances their usability. This method enables reliable and efficient analysis of experimental data for single subject and multi-person synchronization. In a previous paper (Jacoby et al., 2015), we showed how to significantly reduce the estimation error and eliminate the bias of parameter estimation methods by adding a simple and empirically justified constraint on the parameter space. By applying this constraint in conjunction with the tools of matrix algebra, we here develop a novel method for estimating the parameters of most linear models described in the literature. Through extensive simulations, we demonstrate that our method reliably and efficiently recovers the parameters of two influential linear models: Vorberg and Wing (1996), and Schulze et al. (2005), together with their multi-person generalization to ensemble synchronization. We discuss how our method can be applied to include the study of individual differences in sensorimotor synchronization ability, for example, in clinical populations and ensemble musicians.

Keywords

Sensorimotor synchronization, linear models, phase correction, period correction, ensemble synchronization, generalized least squares

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1. Introduction

Sensorimotor synchronization is the temporal coordination of a rhythmic movement with an external rhythm. This ubiquitous behavior has been studied by measuring the performance of musicians or dancers, or more commonly in the laboratory in the form of finger-tapping experiments in which subjects are instructed to tap along with simple rhythmic sequences or controlled adaptive stimuli (for a review, see Repp, 2005; Repp & Su, 2013). There are a number of approaches for modeling sensorimotor synchronization, each focusing on different aspects of this remarkably intricate behavior. One of the leading methods is the class of 'event-based models', which attempt to predict the onset of the next response (i.e., the time the finger impacts upon the tapping surface) from the previous onsets of the stimulus and responses (Hary & Moore, 1987a,b; Mates, 1994a,b; Michon, 1967; Pressing, 1998a, b; Pressing & Jolley-Rogers, 1997; Schulze & Vorberg, 2002; Schulze et al., 2005; Thaut et al., 1998; Vorberg & Schulze, 2002; Vorberg & Wing, 1996; Wing & Kristofferson, 1973). Other approaches emphasize nonlinear aspects of synchronization by using coupled oscillator models (Large & Jones, 1999; Large et al., 2002; McAuley & Jones, 2003; Torre et al., 2010) or time-series analysis of fractal long-term correlations in the stimulus and response sequence (Chen et al., 2002; Delignières et al., 2009; Ding et al., 2002; Torre & Balasubramaniam, 2009; Torre & Delignières, 2008a, b; Wing et al., 2004).

The focus of the current article is on linear event-based models. These are generally constructed to model sensorimotor and cognitive processes involved in synchronization (e.g., temporal error correction), while accounting for different sources of biological noise in the nervous system (e.g., motor variance associated with central vs. peripheral processing). The parameters of these models are of interest because they represent the relative engagement of the aforementioned processes. Note also that these parameters are essential for measuring the goodness of fit required for the evaluation of the model assumptions.

The current article is a follow-up to a previous paper (also published in this special issue: Jacoby et al., 2015), where we pointed out that one of the most influential insights into the modeling of sensorimotor synchronization was made in the context of the synchronization-continuation paradigm. In this simple paradigm, the base tempo is initially given to the subject by an external metronome, which then stops, while the subject is required to respond by continuing to tap a finger at the given tempo. Seminal modeling work on data from the continuation phase of such an experiment led Wing and Kristofferson (1973) to propose that two different noise components contribute to the overall variability of tapping: the variability of an internal timekeeper (σ_T^2) and the variability of motor delays (σ_M^2), where the latter reflects the inherent inaccuracies of the motor system. On this basis, Wing and Kristofferson (1973) suggested that the

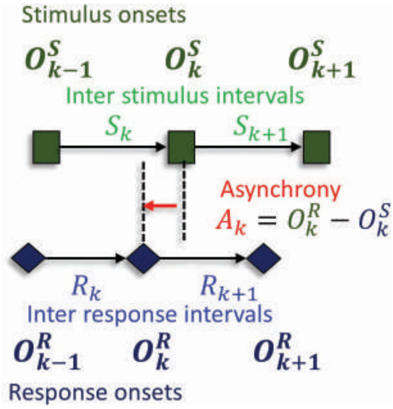

Figure 1. O^R_k and O^S_k are the onsets of the response and stimulus at beat k. $S_k = O^S_k - O^S_{k-1}$ and $R_k = O^R_k - O^R_{k-1}$ are the inter-stimulus and inter-response intervals, respectively. $A_k = O^R_k - O^S_k$ is the asynchrony. This figure is published in color in the online version.

next inter-response interval at beat number $k+1$, R_{k+1} takes the form of the following equation:

$$R_{k+1} = T_k + M_{k+1} - M_k \tag{1}$$

where T_k and M_k are the timekeeper and motor noise at beat number k. Wing and Kristofferson (1973) further assumed that T_k and M_k are independent, and that the mean of T satisfies: $E(T) = \tau$, where τ is the experiment base tempo. Note that in practice, the timekeeper variance was always found to be larger — usually by a substantial margin — than the motor variance (for a review, see Wing, 2002).

Later work extended the Wing and Kristofferson model to sensorimotor synchronization, specifically to a scenario where the subject synchronizes with an isochronous metronome (Vorberg & Wing, 1996). Let us denote by O^R_k and O^S_k the onsets of the response and stimulus onsets. We further denote by R_k, S_k, and A_k the inter-response interval, inter-stimulus interval and asynchrony, respectively. Namely (see also Fig. 1):

$$R_k = O^R_k - O^R_{k-1};\ S_k = O^S_k - O^S_{k-1};\ A_k = O^R_k - O^S_k \tag{2}$$

Vorberg and Wing (1996) and Vorberg and Schulze (2002) suggested the following linear phase correction model[1]:

$$A_{k+1} = (1 - \alpha)A_k + T_k + M_{k+1} - M_k - S_{k+1} \tag{3}$$

[1] Note the small difference between this notation and metronome intervals C_n as defined in Vorberg and Schulze (2002): $C_n = O^S_{n+1} - O^S_n = S_{n+1}$.

which supplements Wing and Kristofferson's noise term with a hypothesized phase correction process controlled by parameter α (phase correction constant). The same model can be used for sequences of stimuli with occasional timing perturbations (slight shifts in time of stimulus onsets), as long as the general tempo of the stimulus (the mean of S) remains constant (Repp et al., 2012).

Vorberg and Schulze (2002) analyzed the performance of an estimation procedure in which moments are estimated from empirical data (Pearson, 1896) and used to fit the parameters of the model. The fit is done based on the match between the model's predictions and the empirical auto-covariance function (*acvf*) estimates. This method works also for metronomes whose intervals vary stochastically (independent and identically distributed random variables). However, if the metronome sequence has a different structure, e.g., a constant metronome with occasional abrupt phase changes as used in Repp et al. (2012), there is no simple analytical solution. In this case the estimation process is relatively slow, requiring the fitting of the *acvf* of the empirical asynchronies to the one computed from simulations.

When discussing this method, Vorberg and Schulze (2002) indentified a problem known as parameter interdependence (Li et al., 1996), which limits the accuracy of estimation procedure of this method, sometimes even rendering it impossible to use (Vorberg & Schulze, 2013). In a companion to the current article (Jacoby et al., 2015), we analyzed the problem. We used the Cramér–Rao lower bound (Cramér, 1999; Rao, 1992)—henceforth CRLB, a well-known lower bound on estimation accuracy, to show that any method of estimating the parameters of the linear phase correction model (Vorberg & Wing, 1996) may be prone to failure unless further assumptions are made. We then proposed a solution to this problem by showing that a reliable estimate can be obtained by adopting the simple assumption that the motor variance is smaller than the timekeeper variance[2]:

$$\sigma_M^2 < \sigma_T^2 \tag{4}$$

We showed that a modified version of the moment estimation method that applies this assumption no longer suffers from parameter interdependence, and provides an unbiased estimator of phase correction and timekeeper and motor noise variance.

[2] Figure 1 of Jacoby et al. (2015) provides an explanation why constraining the parameter space could dramatically reduce estimation errors. A detailed analysis of this issue can be found in the classical works of Cramér and Rao (Cramér, 1999, originally 1946; Rao, 1992, originally 1945). Intuitively, the interdependence comes from points in the parameter space that do not satisfy eqn. (4). When these points are not considered for parameter estimation, the problem disappears and the original method performs well for all cases. This is explained in detail in Jacoby et al. (2015).

In the current study we present an alternative technique, termed the bounded Generalized Least Squares method (henceforth bGLS), to compute the model parameters. The bGLS method is based on rewriting synchronization models in matrix notation. This re-formalization affords using linear algebra techniques for an efficient iterative algorithm. bGLS has a number of advantages: it works for any type of synchronization sequence, and is faster and more reliable than previous methods. The bGLS method is based on standard generalized least square regression methods (Aitken, 1935). Note, however, that conventional methods of parameter estimation implemented in standard data analysis software (e.g., MATLAB[3] or R), will not work here, because the CRLB limits any estimation procedure. Our bGLS method takes into account a specific assumption that can only be made in the context of sensorimotor synchronization, namely, that the motor variance is smaller than the timekeeper variance, and therefore achieves reliable estimates.

After we outline the application of the bGLS algorithm to standard sensorimotor synchronization paradigms, we show how it generalizes to the case of an ensemble of synchronizers maintaining a base tempo, a model similar to the recent string quartet analysis presented by Wing et al. (2014). We thereby describe how the bGLS method enables the efficient extraction of phase coupling as well as individual timekeeper and motor noise estimates.

Thus far we have confined our discussion to synchronization with sequences that are isochronous or contain occasional timing perturbations. Many real-life situations require individuals to synchronize with sequences that change tempo. In music, for example, performers introduce tempo changes to communicate information about musical structure and their expressive intentions (Desain & Honing, 1994; Janata et al., 2012; Repp, 1990, 1995; Repp & Bruttomesso, 2009). To address sensorimotor synchronization with tempo changes, a number of researchers (e.g., Mates, 1994a, b; Schulze et al., 2005; Semjen et al., 1998; Thaut et al., 1998) have suggested that the timekeeper tempo can be adjusted by a hypothetical cognitive process involving period correction, i.e., a compensatory change in the timekeeper period. Repp and Keller (2004) found that cognitive load modulates period correction, but has no effect on phase correction. This led to the suggestion that phase correction is automatic and immediate, whereas period correction requires attention and explicit control. Further experiments have shown that phase correction is engaged by subliminal phase shifts, providing additional support to the automaticity of this process (Madison & Merker, 2004; Repp, 2001).

Schulze et al. (2005) presented a model that takes period correction into account.[4] The core of this model is identical to the linear phase correction model

[3] In MATLAB, the ARMA model estimator is implemented by the *armax* function.

[4] Note that in a previous work (Jacoby & Repp, 2012), we analyzed several period correction models (Hary & Moore, 1987a, b; Mates, 1994a, b; Michon, 1967; Schulze et al., 2005) and showed that even though they are based on different theoretical principles they are mathematically equivalent. However, we did not address the structure of correlations (apart from one model described in Schulze et al., 2005, that

(Vorberg & Schulze, 2002; Vorberg & Wing, 1996), but the model further assumes that the mean of the timekeeper t_k changes according to the following formulae:

$$A_{k+1} = (1 - \alpha)A_k + T_k + M_{k+1} - M_k - S_{k+1} \tag{5}$$

$$t_k = t_{k-1} - \beta A_k \tag{6}$$

for $k = 0, 1, 2, \ldots$ and where it is assumed that t_0 is the base tempo.

Note that this differs from the previous model by including an unobserved timekeeper mean (t_k). In this paper, we show how to derive an equivalent form of this model that does not require the rendering of unobserved variables. We then discuss how to generalize the bGLS method to estimate the period correction model's parameters, and present simulation results indicating that it can reliably extract these parameters.

This article concludes with descriptions of extensions to multi-person synchrony and synchronization with hierarchically structured temporal sequences, such as those associated with musical meter. To handle multi-person synchrony, we develop an ensemble synchronization model with period and phase correction, using simulations to prove the tractability of our novel approach for this complex scenario. With regard to hierarchical structured sequences, we demonstrate how to compute metrically dependent phase correction parameters where different phase correction constants are applied for different positions in the hierarchical metrical structure.

The main goal of the work described in this article is to extend the available analytic toolbox for modeling sensorimotor synchronization by providing a unified, fast, and reliable method for parameter estimation from the experimental data of single person and ensemble synchronization. This paper is accompanied by a freely available MATLAB code[5], to be used in practical applications for parameter estimation.

2. The bGLS Method for Estimating the Parameters of Phase Correction Models

Here we explain the rationale behind the bGLS method (briefly described in the appendix of Repp et al., 2012) for estimating the parameters of phase correction models. Note that this method can work both in the case of isochronous and non-isochronous pacing sequences.

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was fully analyzed in the appendix of the paper). This effectively means that we assumed uncorrelated residual noise (as in the Michon, 1967, and Mates, 1994a, model, but contrary to Hary & Moore, 1987a, b, and Schulze et al., 2005). The current work was motivated by the desire to find a reliable estimation procedure that takes into account the specific correlation structure proposed by each of these models.

[5] The code is available at *http://dx.doi.org/10.6084/m9.figshare.1391910*.

2.1. Approximated Maximum Likelihood Estimation

The motivation behind the bGLS method is to find an approximated Maximum Likelihood Estimator. In many classes of problems (including ARMA models), this kind of estimator is reliable and unbiased, at least when sample size is sufficiently large (Ljung, 1998).

Let:

$$\begin{aligned}
y &= [A_1 + S_1 - E(A+S), A_2 + S_2 - E(A+S), \ldots, A_N + S_N - E(A+S)]^T,\\
B &= [A_0 - E(A), A_1 - E(A), \ldots, A_{N-1} - E(A)]^T,\\
Z &= [H_0, H_1, \ldots, H_{N-1}]^T,\\
x &= 1 - \alpha.
\end{aligned}$$

where: $H_k = T_k + M_{k+1} - M_k - E(T)$, and M^T is henceforth used to denote the transpose of a matrix or a vector M, and M^{-1} is the inverse matrix of a matrix M.

Then the linear phase correction model can be written in matrix notation as:

$$y = Bx + Z \tag{7}$$

where Z is a multivariate Gaussian noise[6] with the following *acvf* ($\gamma_Z(j) \equiv \mathrm{Cov}$ $(Z(k+j), Z(k))$):

$$\gamma_Z(0) = 2\sigma_M^2 + \sigma_T^2,\ \gamma_Z(1) = -\sigma_M^2 \text{ and } \gamma_Z(j) = 0 \quad \text{for } j > 1 \tag{8}$$

Note that Z has zero mean and an N-by-N covariance matrix Σ of the form:

$$\Sigma = \gamma_Z(0)I + \gamma_Z(1)\Delta \tag{9}$$

Here I is the N-by-N identity matrix and Δ is an N-by-N matrix with ones on the two secondary diagonals and 0 elsewhere:

$$\Delta = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 0\\
1 & 0 & 1 & \ddots & \vdots & 0\\
0 & 1 & 0 & \ddots & 0 & \vdots\\
\vdots & 0 & \ddots & 0 & 1 & 0\\
0 & \vdots & \ddots & 1 & 0 & 1\\
0 & 0 & \cdots & 0 & 1 & 0
\end{bmatrix}$$

[6] Several previous studies used Gamma noise as part of the estimation procedures or the model. Using simulations, we verified that the parameters of a model randomized with Gamma noise can be estimated if we assume Gaussian distribution.

In other words, Σ has the following form:

$$\begin{aligned}\Sigma = \mathrm{Cov}(Z) &= \begin{bmatrix} \sigma_T^2 + 2\sigma_M^2 & -\sigma_M^2 & & 0 & 0 \\ -\sigma_M^2 & \ddots & & \ddots & 0 \\ 0 & \ddots & & \ddots & -\sigma_M^2 \\ 0 & 0 & & -\sigma_M^2 & \sigma_T^2 + 2\sigma_M^2 \end{bmatrix} \\ &= \begin{bmatrix} \gamma_Z(0) & \gamma_Z(1) & & 0 & 0 \\ \gamma_Z(1) & \ddots & & \ddots & 0 \\ 0 & \ddots & & \ddots & \gamma_Z(1) \\ 0 & 0 & & \gamma_Z(1) & \gamma_Z(0) \end{bmatrix}\end{aligned} \tag{10}$$

Since we assume that Z is a multivariate Gaussian variable with 0 mean and an N-by-N covariance matrix Σ, the likelihood (equal to the probability density function) of Z given the parameters x, Σ is given by the following formula (Feller, 2008):

$$\begin{aligned}\mathrm{Prob}(Z|x,\Sigma) &= \mathrm{Prob}(y - Bx\,|x,\Sigma) \\ &= \frac{1}{\sqrt{(2\pi)^N|\Sigma|}}\, e^{-\frac{1}{2}(y - Bx)^T\left(\Sigma^{-1}\right)(y - Bx)}\end{aligned} \tag{11}$$

and the log of the likelihood is given by the following formula:

$$\mathrm{LL}(Z|x,\Sigma) = -\frac{N}{2}\log(2\pi) - \frac{1}{2}\log(|\Sigma|) - \frac{1}{2}(y - Bx)^T\left(\Sigma^{-1}\right)(y - Bx) \tag{12}$$

One candidate for estimating the parameters would therefore be to find the parameters that maximize the likelihood (or the log-likelihood):

$$\left(a^{\mathrm{ML}}, \gamma_0^{\mathrm{ML}}, \gamma_1^{\mathrm{ML}}\right) = \mathrm{argmax}x_{\alpha,\gamma_0,\gamma_1}\, LL(Z|x = 1 - \alpha, \Sigma = \gamma_0 I + \gamma_1 \Delta) \tag{13}$$

It is nontrivial to find an efficient algorithm that maximizes the likelihood, mainly because the error correction parameter and the covariance matrix are unknown. However, it is relatively simple to estimate the optimal parameters when either Σ or x are known.

2.2. Cases Where Σ or x Are Known

When the covariance matrix Σ is known, the log-likelihood is maximized if the term: $(y-Bx)^T(\Sigma^{-1})(y-Bx)$ is minimized. The solution is known as the General

Least Squares (GLS) and is given by the following formula (Lawson & Hanson, 1974)[7]:

$$x^{\mathrm{GLS}} = \left(B^T \Sigma^{-1} B\right)^{-1} \left(B^T \Sigma^{-1}\right) y \tag{14}$$

When x is known and Σ is unknown, and N is large, the log-likelihood is maximized when:

$$\Sigma = \gamma_0^{\mathrm{GLS}} I + \gamma_1^{\mathrm{GLS}} \Delta \tag{15}$$

When γ_0^{GLS}, γ_1^{GLS} are computed as follows:

$$d = y - Bx \tag{16}$$

$$\gamma_0^{\mathrm{GLS}} = \frac{1}{N} \sum_{k=1}^{N} d(k) d(k), \gamma_1^{\mathrm{GLS}} = \frac{1}{N-1} \sum_{k=1}^{N-1} d(k+1) d(k) \tag{17}$$

This, of course, simply corresponds to estimating the *acvf* (Feller, 2008).

The problem is that in our case (eqn. 13), both x and Σ are *unknown*. However, it is possible to circumvent this problem by performing alternating iterations: this involves maximizing the likelihood by changing x when Σ remains fixed (using eqn. 14) and then fixing x and maximizing the likelihood according to Σ (using eqns 16–17). As this process is repeated multiple times, at every step the likelihood increases. This process terminates when convergence, i.e., a local maximum, is reached. This method is a variant of the so-called 'double projection' method (Boyd & Vandenberghe, 2004), which entails projecting alternately into two convex sets, and is also guaranteed to converge to a local maximum of the likelihood function. One can show that under asymptotic conditions, this procedure actually converges to the global maximum (Ljung, 1998), thereby efficiently solving eqn. (13).

It should be noted that using this method directly would *not* work; it will perform similarly to the un-bounded *moments-acvf method* (see Jacoby et al., 2015). This is depicted in Fig. 2. Even though the unbounded bGLS method is less biased than the un-bounded *acvf method*, both methods have extremely large estimation errors. This is because they are expected to suffer from the limitation of the CRLB described in Jacoby et al. (2015), where it is shown that any estimation method will have large estimation error if no further constraints are assumed.

[7] When Σ is diagonal (in our case when $\sigma_M^2 = 0$), the solution is obtained when $||y - Bx||^2$ is minimized, which is in fact the least square solution (Lawson & Hanson, 1974; Strang, 2006). In this case: $x^{\mathrm{LS}} = (B^T B)^{-1} (B^T) y$.

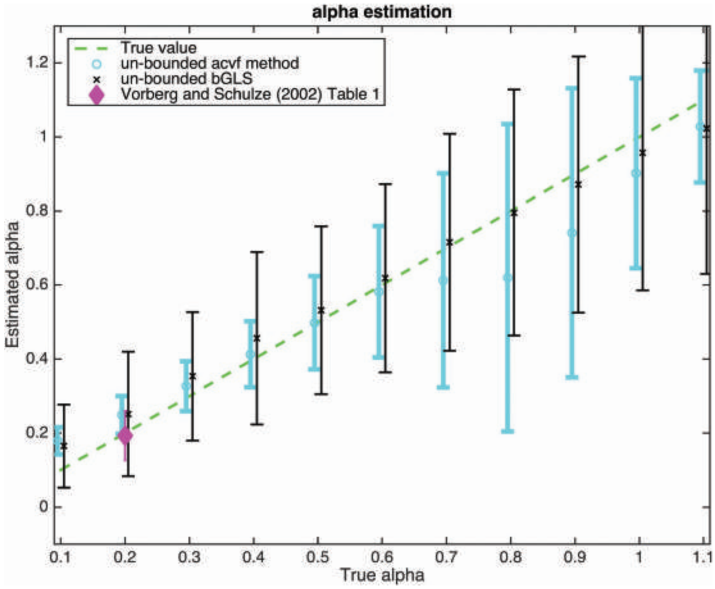

Figure 2. Comparison of the unbounded *acvf* method [circles] and the unbounded Generalized Least Squares method [thin line]. The correct underlying parameters are plotted as the dashed line. The figure shows the mean estimates based on 100 simulated renditions of the phase correction model of eqn. (3), for different α values (x-axis) with parameters $\sigma_T^2 = 100$, $\sigma_M^2 = 25$, $nseq = 15$, and $N = 30$. Error bars represent the standard deviation of the estimates. Both methods show extremly large estimation errors for large α values. This demonstrates the necessity of using further constraints in order to obtain a reliable estimates. This figure is published in color in the online version.

2.3. *bGLS Method for Phase Correction Models*

The limitation presented in Fig. 2 can be remediated by restricting the parameter space as in the bounded *acvf* method (see Jacoby et al., 2015). This can be done by modifying the estimates of the residual error moments in every iteration of the Aitken (1935) procedure. Intuitively, this step reduces the ambiguity in the likelihood function (see Fig. 1 of Jacoby et al., 2015)

This is formalized in the bGLS algorithm.

2.3.1. *The bGLS Algorithm*

- Input: *nseq* sequences of $N+1$ asynchronies $\{A_{k=0\ \ldots\ N}\}_{i=1\ \ldots\ nseq}$ and inter-stimulus intervals $\{S_{k=0\ \ldots\ N}\}_{i=1\ \ldots\ nseq}$.
- *niter*, a constant that determines the number of iterations[8],

Output: parameter estimates $(\overline{\alpha}, \overline{\sigma}_T, \overline{\sigma}_M)$

Preliminary steps:

[8] $niter = 20$ in all the simulations reported in this paper.

0. Compute the grand average $E(A)$ and $E(S)=\tau$ from all the data[9],
1. For each sequence of the *nseq* sequences compute:

$$y=\left[A_1+S_1-E(A+S), A_2+S_2-E(A+S), \ldots, A_N+S_N-E(A+S)\right]^T,$$
$$B=\left[A_0-E(A), A_1-E(A), \ldots, A_{N-1}-E(A)\right]^T.$$

Main steps:

2. For each sequence of the *nseq* sequences, compute:
   2.1. Start by setting $\Sigma_1=I$ (the N-by-N identity matrix).
   2.2. Iterate the following equations:
      (i) Set: $x_n=\left(B^T\Sigma_n^{-1}B\right)^{-1}\left(B^T\Sigma_n^{-1}\right)y$
      (ii) Set: $d_n=y-Bx_n$
      (iii) Set: $\gamma_n(0)=\frac{1}{N}\Sigma_{k=1}^{N}d_n(k)d_n(k)$
      (iv) Set: $\gamma_n(1)=\frac{1}{N-1}\Sigma_{k=1}^{N-1}d_n(k+1)d_n(k)$
      (v) Adjust $\gamma_n(1)$ by decreasing or increasing it so that:

$$0<-\gamma_n(1)<\gamma_n(0)+2\gamma_n(1).$$

      (vi) Set: $\Sigma_{n+1}=\gamma_n(0)I+\gamma_n(1)\Delta$, when I is the N-by-N identity matrix and Δ is an N-by-N matrix with one on the two secondary diagonals and 0 elsewhere.

   2.3. At the last iteration of each sequence set:

$$\overline{\alpha}=1-x_{niter}, \overline{\sigma}_M=\sqrt{-\gamma_{niter}(1)}, \overline{\sigma}_T=\sqrt{\gamma_{niter}(0)-2\overline{\sigma}_M^2}.$$

3. Take the average values obtained from the *nseq* sequences.

The bGLS method is usually much faster than the *acvf* method (or the bounded *acvf* method). For parameters ($\alpha=0.2$, $\sigma_T^2=100$, $\sigma_M^2=25$, $N=30$, $nseq=1$), the bounded *acvf* method took 0.12 s whereas the bGLS method took 0.0032 s[10],

[9] It is important to compute the mean from as much available data as possible. Inaccuracy in computing the mean due to lack of data can cause degraded performance. Note that this method only works when the tempo is relatively fixed. We derive a formula for sequences with changing tempos in Sect. 4 of this paper.

[10] All algorithms in this paper were implemented efficiently using MATLAB on an Intel i7 Windows 64 system. Running times are highly dependent on parameters, implementation and hardware. All running times provided in the paper were computed using the same hardware and are given to show the general relation between running times of the different algorithms.

an improvement of a factor of more than 30. Note that the efficency of this algorithm enables the parameters to be computed online, which may be important when designing experiments with adaptive metronomes (e.g., Fairhurst et al., 2014; Repp & Keller, 2008; Vorberg, 2005) or for applications in human–machine interaction.

Jacoby et al. (2015) compared bounded *acvf* method to the bGLS method (see Jacoby et al., 2015, Fig. 7). It was shown that the bGLS method outperforms the alternative method, and provides an unbiased estimation procedure. Note that this advantage depends on the constraint of eqn. (4), namely, that motor variance is smaller than timekeeper variance.

The main advantage of the bGLS method is that it can be used for non-isochronous sequences directly without change. Figure 3 shows simulation

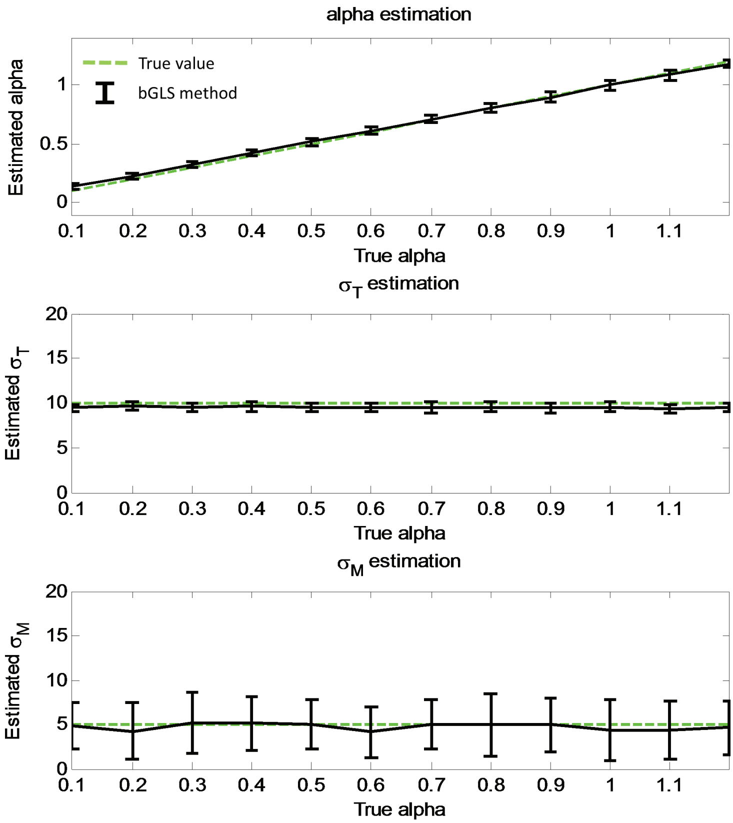

Figure 3. The bGLS method with a non-isochronous sequence. Ideal performance is marked in dashed line. We simulated 1000 iteration for different α values (x-axis) with parameters ($\sigma_T^2 = 100$ and $\sigma_M^2 = 25$, $nseq = 15$, $N = 30$). The top, middle, and bottom graphs show the mean and standard deviation of the estimates for phase correction, timekeeper variance, and motor variance, respectively. This figure is published in color in the online version.

results with a metronome containing perturbations, taken from a phase correction experiment by Repp et al. (2012). The close fit to the true model parameters indicates that the method works very well in this case.

2.4. *The Relationship between Standard Regression and the Linear Phase Correction Model*

Note that when $\sigma_T \gg \sigma_M$, the model:

$$A_{k+1} + S_{k+1} = (1 - \alpha)A_k + T_k + M_{k+1} - M_k \tag{18}$$

can be approximated by the following model:

$$A_{k+1} + S_{k+1} = (1 - \alpha)A_k + V_k \tag{19}$$

where V_k is Gaussian white noise, namely, $\gamma_V(j) = 0$ for $j \geq 1$. We denote this the 'white noise' model.

For this model, we can simply use the well-known Least Squares method (Lawson and Hanson, 1974) for estimating α; namely:

$$1 - \alpha = \left(B^T B\right)^{-1} B^T y \tag{20}$$

Note that $(B^T B)^{-1} B^T y \approx \gamma_A(1)/\gamma_A(0)$.

As in Wing (2002), we expect the motor variance to remain constant, whereas the timekeeper variance increases with inter-onset interval duration from about 100 to about 400 ms when the inter-onset intervals range between 220 to 490 ms. Thus we expect that when we increase the inter-onset interval, the Least Squares solution will become increasingly accurate.

The simulations shown in Fig. 4 illustrate the outcomes of applying the Least Squares solution when $\sigma_M^2 = 25$ and $\sigma_T^2 = 100, 300, 500$ (Fig. 4 top). Note that with high timekeeper variance the Least Squares solution performs surprisingly well. Comparing the performance of the bounded *acvf* method (Fig. 4 bottom), confirms that the Least Squares method is biased, but this bias decreases with increasing σ_T^2. Furthermore, the Least Squares estimate seems consistent, whereas the *acvf* method shows larger estimation error but less bias. In cases where the aim is not to find the true value of α but rather to focus on individual differences, it would be appropriate to use the robust white noise model with the simple Least Squares solution instead of the linear phase correction model (Vorberg & Wing, 1996). This might be particularly advisable for relatively large inter-onset intervals (slow tempi) where the linear phase correction model approaches the white noise model (Repp, 2011). This might also retrospectively justify the choice of the white noise model proposed by Wing et al. (2014).

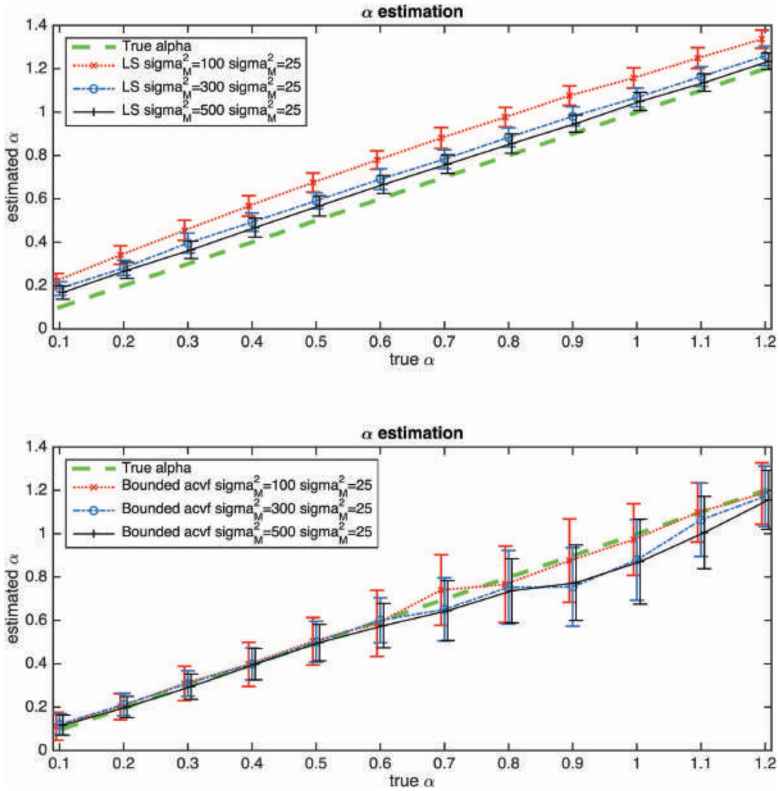

Figure 4. Comparison of mean and standard deviation of the estimations based on motor noise and least square estimation (LS) that ignores motor noise. Motor variance was fixed ($\sigma_M^2 = 25$), and the different lines present different $\sigma_T^2 = 100, 300, 500$ timekeeper variance. In all simulations $nseq = 15$ and $N = 30$. This figure is published in color in the online version.

3. Phase Correction for Ensemble Synchronization

In recent years there has been increasing interest in multi-person synchronization (Butterfield, 2010; Darabi et al., 2008; Goebl & Palmer, 2009; Keller, 2008; Keller & Appel, 2010; Loehr & Palmer, 2011; Marchini et al., 2012; Moore & Chen, 2010; Rasch, 1988; Shaffer, 1984; Wing et al., 2014). Scenarios that have been studied in the laboratory include two persons synchronizing simple movements (e.g., taps or claps) with each other (Darabi et al., 2008), or with an artificial metronome simulating a human partner (Fairhurst et al., 2014; Repp & Keller, 2008).

Research with piano duos (Goebl & Palmer, 2009; Loehr & Palmer, 2011) and dyadic finger tapping tasks (Konvalinka et al., 2010; Merker et al., 2009) has found that compensatory adjustments associated with phase correction lead to co-dependencies in which successive time intervals produced by two interacting individuals are similar in duration. Furthermore, a recent study that employed a task requiring paired musicians to tap in alternation with an isochronous pacing signal found similarities between successive asynchronies produced by alternating individuals' taps relative to the pacing tones (Nowicki et al., 2013). This may represent mutual temporal assimilation in the use of phase correction, perhaps constituting a form of behavioral mimicry that facilitates interpersonal coordination in situations such as ensemble performance.

Another interesting area is related to the modeling of a group of synchronizers attending to one another in musical ensembles (for a review see Keller, 2014), as in the case of string quartet performances analyzed by Wing et al. (2014). These authors found intricate patterns of phase correction when they examined timing dependencies between all possible pairs of performing musicians. In this case the musicians played at a relatively fixed tempo, similar to the assumptions in previous sections of this paper.

Our methodology can easily be adapted to handle such cases. We start by rewriting the linear phase correction model in the following form:

$$R_{k+1} = -\alpha A_k + H_k \tag{21}$$

where: $H_k = T_k + M_{k+1} - M_k; M_k \sim N\left(0, \sigma_M^2\right); T_k \sim N\left(0, \sigma_T^2\right)$

Note that without loss of generality, we reduced the empirical mean:

$$R_{k+1} \rightarrow R_{k+1} - E(R) \text{ and } A_k \rightarrow A_k - E(A)$$

In practice we will reduce the empirical mean from the computation before we start (similarly as in the preliminary steps of the bGLS algorithm).

This model could be easily extended to multi-person cases as illustrated in Fig. 5.

Figure 5 schematically shows the onsets O_k^R of a given synchronizer and the onsets of the other P participants: $O_k^{S_1}, \ldots, O_k^{S_P}$. We denote by $A_k^1, \ldots, A_k^P$ the asynchronies[11] of the synchronizer compared to the other participants $A_k^i = O_k^R - O_k^{S_i}$.

The multi-person model is therefore:

$$R_{k+1} = \sum_{i=1}^{P} -\alpha_i A_k^i + H_k \tag{22}$$

and α_i are the phase correction coupling constants with all other participants, and H_k is as in eqn. (21).

[11] Note that as a preprocessing step, asynchronies larger than a certain threshold are identified and replaced by the mean. For example, for an inter-stimulus interval of about 500 ms, a reasonable threshold would be ~200 ms.

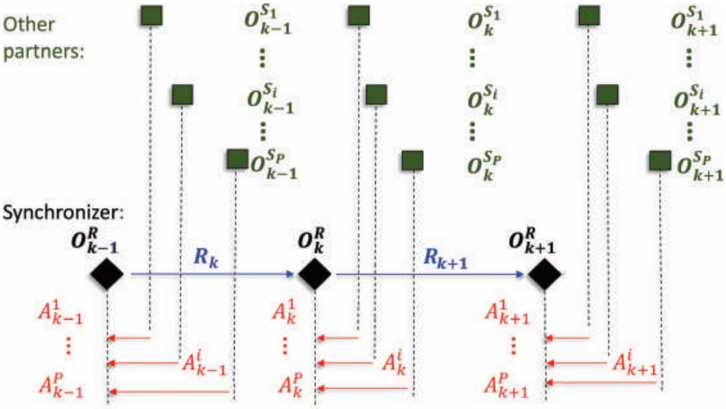

Figure 5. Schematic illustration of onsets for synchronizing in multi-person cases. We denote the onsets of the responses of the current synchronizer as O_k^R. We denote by $O_k^{S_i}$ the onsets of the P other partners. The asynchronies between response and partner i are given by A_k^i. This figure is published in color in the online version.

This model can be written in matrix notation as:

$$y = Bx + Z \tag{23}$$

where:

$$B = \begin{bmatrix} A_0^1 - E(A^1) & \cdots & A_0^P - E(A^P) \\ \vdots & & \vdots \\ A_{N-1}^1 - E(A^1) & \cdots & A_{N-1}^P - E(A^P) \end{bmatrix}; y = \begin{bmatrix} R_1 - E(R) \\ \vdots \\ R_N - E(R) \end{bmatrix}; Z = \begin{bmatrix} H_0 \\ \vdots \\ H_{N-1} \end{bmatrix}; x = \begin{bmatrix} -\alpha_1 \\ \vdots \\ -\alpha_P \end{bmatrix} \tag{24}$$

Using the bGLS method, these parameters can be easily extracted from experimental data. We can apply the bGLS algorithm without any changes other than replacing y, B, and x with eqn. (24).

To demonstrate that we can recover the real parameters from this model, we simulated data based on the Wing et al. (2014) paper. The dataset comes from a scenario where two separate string quartets played a relatively regular rhythmic musical piece while the onsets of their sounds were measured. Since the musical score had the same rhythmic units (apart from an ornament at the end of the first violin part), and the musicians maintained an approximately fixed tempo, it was possible to use the phase correction model. Wing et al. (2014) did not use the full model, but rather chose a form that is the multi-person generalization of the white noise model of Sect. 2.4:

$$R_{k+1} = \sum_{i=1}^{P} -\alpha_i A_k^i + V_k \tag{25}$$

Here V_k is Gaussian independent and identically distributed noise. Effectively, this is equivalent to assuming that the motor noise is extremely small, which may be reasonable in the case of expert performers. Section 2.4 contains a discussion comparing this model and the fully coupled noise model for the case of a single synchronizer. However, we can recover the coupling parameters with the bGLS even in the presence of larger motor variance.

Figure 6 shows 100 iterations of a simulated string quartet. The true values of phase correction parameters among all the players are given in the bars in the upper diagram: note that there are 12 phase correction parameters. In the lower diagram, the bars indicate the timekeeper and motor variances of the simulated quartet. As can be seen, the method can be used to recover both the noise and coupling constants with relatively small error. Note that this demonstrates the robustness of our novel approach, since trying to estimate the full model parameters with other methods has been thus far unsuccessful (Vorberg & Schulze, 2013).

Figure 7 presents a reanalysis of the data published by Wing et al. (2014). We computed the coupling phase constants together with motor and timekeeper variance. Our results qualitatively match the results of Wing et al. (2014). For example, the coupling constants were not much larger than 0.4, and usually either very small (near 0). However, since the motor variance we computed was not small, the results here are not identical to the results in Fig. 6 in Wing et al. (2014), who did not estimate the motor variance.

The running time of computing all phase and noise parameters for one trial of this experiment was only 0.075 s (the full results presented in Fig. 7, which contain data from 32 separate recordings, were completed in about 2.4 s), which in principle allows for the implementation of the method in real-time applications of multi-person synchronization (e.g., one person interacting with multiple virtual partners simultaneously).

4. The Schulze et al. (2005) Period Correction Model

4.1. Rewriting Period Correction Models without Unobserved Variables

Schulze et al. (2005) suggest an extension to the linear phase correction model to handle synchronization with tempo-changing sequences. In cases where the internal timekeeper is not constant, this model has the same structure:

$$A_{k+1} = (1-\alpha)A_k + T_k + M_{k+1} - M_k - S_{k+1} \tag{26}$$

But now T_k is distributed $N\left(t_k, \sigma_T^2\right)$, and (note that this equation corrects a minor mistake in eqn. 2 of Schulze et al. (2005)):

$$t_k = t_{k-1} - \beta A_k \tag{27}$$

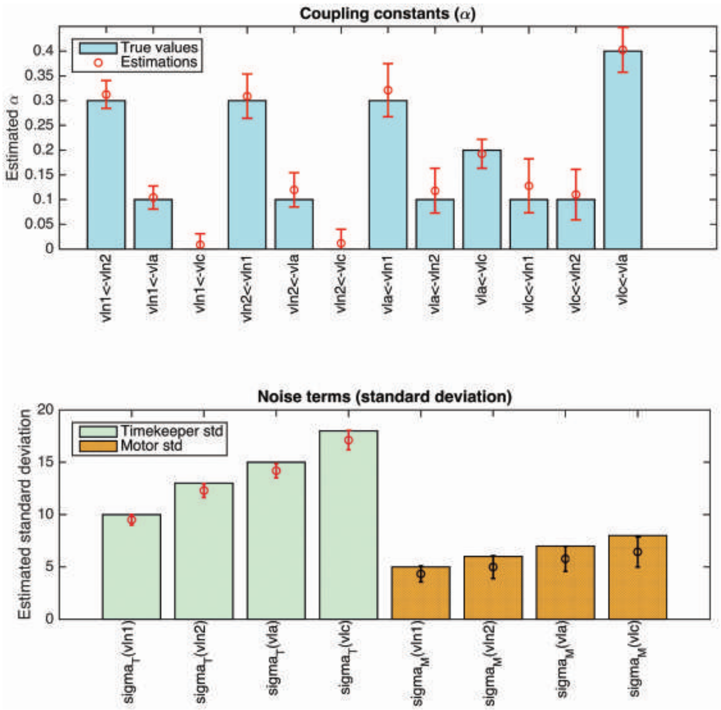

Figure 6. Simulated string quartet with a setting similar to the data in Wing et al. (2014); namely, the same number of synchronizers and the same number of trials and blocks. The top graph shows the phase constants of all four simulated players. The bottom graph shows the different timekeeper standard deviations ($\sigma_T = 10$, 13, 15, and 18 for violin 1, violin 2, viola, and cello, respectively) and motor variances ($\sigma_M = 5, 6, 7, 8$). Estimation of 100 iterations of the simulations with $nseq = 16$, and $N = 40$ are depicted. Error bars represents standard deviation of the estimation error. This figure is published in color in the online version.

The constant β (period correction) determines the update of the internal timekeeper mean. This model assumes that $E(A)=0$, but if this is not the case the empirical mean of A can be reduced $[A \rightarrow A - E(A)]$.

We now show how this can be written in a different form, and subsequently use the bGLS method to compute the period and phase correction constants.

The period correction model can be written therefore as:

$$A_{k+1} + S_{k+1} = (1 - \alpha)A_k + t_k + T'_k + M_{k+1} - M_k \tag{28}$$

where $T'_k \sim N\left(0, \sigma_T^2\right)$, $M_k \sim N\left(0, \sigma_M^2\right)$.

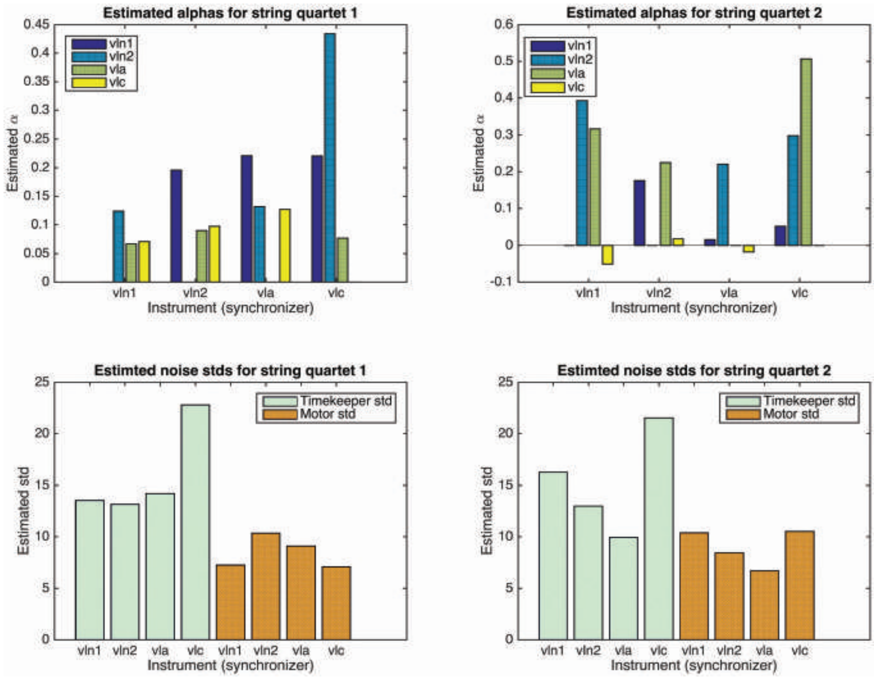

Figure 7. Reanalyis of real string quartets using the data from Wing et al. (2014). The top figures show the estimated coupling constants for quartet 1 (left) and quartet 2 (right). The bottom figures show the estimated timekeeper and motor variance for each of the players. This figure is published in color in the online version.

Now we denote by: $H_k = T'_k + M_{k+1} - M_k$. Then:

$$A_{k+1} + S_{k+1} - A_k = (-\alpha)A_k + t_k + H_k \tag{29}$$

Now since direct computation shows that:

$$A_{k+1} + S_{k+1} - A_k = R_{k+1} \tag{30}$$

Since:

$$A_{k+1} + S_{k+1} - A_k = \left(O^R_{k+1} - O^S_{k+1}\right) + \left(O^S_{k+1} - O^S_k\right) - \left(O^R_k - O^S_k\right) = R_{k+1} \tag{31}$$

Therefore eqn. (29) can be written as:

$$R_{k+1} = (-\alpha)A_k + t_k + H_k \tag{32}$$

Now from eqn. (27):

$$t_k = t_{k-1} - \beta A_k \tag{33}$$

And therefore:

$$t_k = t_{k-1} - \beta A_k = t_0 - \beta A_k - \beta A_{k-1} - \cdots A_1 = T_0 - \beta \Sigma_{j=1}^{k} A_j \tag{34}$$

This implies that eqn. (29) can be written as:

$$R_{k+1} = -\alpha A_k - \beta \Sigma_{j=1}^{k} A_j + H_k + T_0 \tag{35}$$

We denote by:

$$\hat{A}_k \equiv \sum_{j=1}^{k} A_j \tag{36}$$

Therefore eqn. (26) can be written as:

$$R_{k+1} = -\alpha A_k - \beta \hat{A}_k + H_k + T_0 \tag{37}$$

We recall that we applied the transformation $A \longrightarrow A - E(A)$ and $R \longrightarrow R - E(R)$. Therefore, the same model can be written as:

$$R_{k+1} = -\alpha A_k - \beta \hat{A}_k + H_k \tag{38}$$

This model can be also written in matrix notation as:

$$y = Bx + Z \tag{39}$$

where:

$$y = \begin{bmatrix} A_0 - E(A) & \hat{A}_0 - E\left(\hat{A}\right) \\ \vdots & \vdots \\ A_{N-1} - E(A) & \hat{A}_{N-1} - E\left(\hat{A}\right) \end{bmatrix} \tag{40}$$

$$b = \begin{bmatrix} R_1 - E(R) \\ \vdots \\ R_N - E(R) \end{bmatrix}; Z = \begin{bmatrix} H_0 \\ \vdots \\ H_{N-1} \end{bmatrix}; x = \begin{bmatrix} -\alpha \\ -\beta \end{bmatrix} \tag{41}$$

This has exactly the same form of the bGLS algorithm and we therefore apply it without any changes other than replacing y, B, and x with eqns (40) and (41).

4.2. Simulation Results with Period Correction and bGLS

We now report the results of a simulation based on an experiment conducted by Jacoby and Repp (2012). The experiment investigated sensorimotor synchronization with tempo changes involving acceleration and deceleration between slow (ISI $=450$ ms) and fast (ISI $=550$ ms) tempo every 7 to 13 beats.

Figure 8 shows the estimation of the bGLS method. The period correction model was simulated with $\sigma_M=5$, $\sigma_T=10$ with all combinations of $\alpha=$ 0.1, 0.3, 0.5, 0.7, 0.9, 1.1, and $\beta=0.1$, 0.5, 0.9. Each simulated trial had $N=100$ beats. The performance of the method is excellent with extremely small bias and

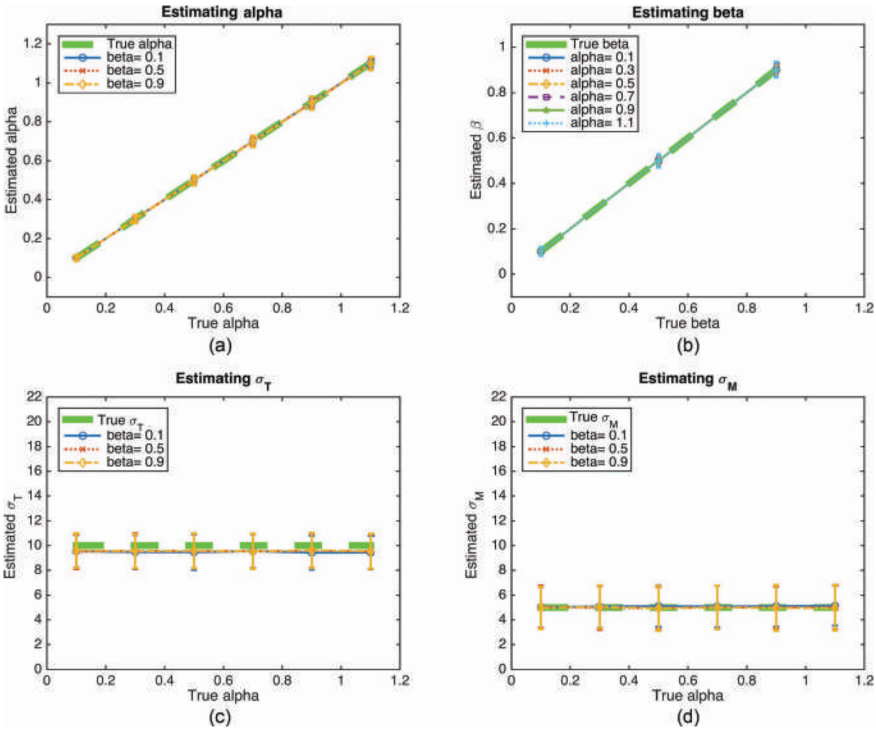

Figure 8. Simulation results for the period correction experiment in Jacoby and Repp (2012). Note that because of the high accuracy of the method there is a large overlap between the simulated results. (a): mean and standard deviation of the estimated α based on 1000 simulations for the three possible β values and six α values. (b): mean and standard deviation of the estimated β based on 1000 simulations for the six possible α values and three β values. (c): mean and standard deviation of the estimated σ_T based on 1000 simulations. (d): mean and standard deviation of the estimated σ_M based on 1000 simulations. This figure is published in color in the online version.

estimation error for period and phase correction parameters and small estimation error of noise magnitudes.

5. Ensemble Synchronization with Phase and Period Correction

5.1. Period and Phase Correction in Matrix Notation

In most instances of multi-person synchrony, the need for period correction would be expected in addition to phase correction. This is the case because some degree of tempo drift can be expected, and, specifically in musical ensemble performance, musicians typically employ aesthetically motivated expressive variations in tempo. We can easily extend our model to handle such cases.

Recall that the single-person case can be written as:

$$R_{k+1} = -\alpha A_k - \beta \hat{A}_k + H_k \tag{42}$$

where $H_k = T'_k + M_{k+1} - M_k$ and $T'_k \sim N\left(0, \sigma_T^2\right)$, $M_k \sim N\left(0, \sigma_M^2\right)$.

We emphasize that, as before, without loss of generality, we reduced the empirical mean:

$$R_{k+1} \rightarrow R_{k+1} - E(R)$$

and

$$A_k \rightarrow A_k - E(A),\ \hat{A}_k \rightarrow \hat{A}_k - E\left(\hat{A}\right)$$

With the same notation as in previous sections we can write:

$$R_{k+1} = \sum_{i=1}^{P} (-\alpha_i) A_k^i + \sum_{i=1}^{P} (-\beta_i) \hat{A}^i{}_k + H_k \tag{43}$$

where α_i and β_i are the phase correction and period coupling constants between the current participant of interest and other participant i, respectively.

And:

$$\hat{A}^i{}_k \equiv \sum_{j=1}^{k} A_j^i \tag{44}$$

is the cumulative sum of the asynchrony with respect of subject i.

Note that the coupling constants are not symmetrical; hence there might be different coupling constants depending on who is the reference subject.

This model can be also written in matrix notation and is therefore solvable with the bGLS method. Again, the algorithm is almost identical to the previous variant of the bGLS method presented in the paper (Sect. 1.3), but the preliminary steps in the algorithm are replaced by computing the matrix B and the vector y as follows.

$$y=\begin{bmatrix} R_1 - E(R) \\ \vdots \\ R_N - E(R) \end{bmatrix} \tag{45}$$

B is the N-by-$2P$ matrix:

$$B=\begin{bmatrix} A_0^1 - E(A^1) & \cdots & A_0^P - E(A^P) & \hat{A}^1{}_0 - E(\hat{A}^1) & \cdots & \hat{A}^P{}_0 - E(\hat{A}^P) \\ \vdots & & \vdots & \vdots & & \vdots \\ A_{N-1}^1 - E(A^1) & \cdots & A_{N-1}^P - E(A^P) & \hat{A}^1{}_{N-1} - E(\hat{A}^1) & \cdots & \hat{A}^P{}_{N-1} - E(\hat{A}^P) \end{bmatrix} \tag{46}$$

Solving the matrix equality:

$$y = Bx + Z \tag{47}$$

Where:

$$x=\begin{bmatrix} -\alpha_1 \\ \vdots \\ -\alpha_P \\ -\beta_1 \\ \vdots \\ -\beta_P \end{bmatrix}$$

And Z is the same as in the single subject variant of eqn. (41).

5.2. *Simulated String Quartet with Period and Phase Correction*

To demonstrate the strength of the bGLS method, we now present the results of a simulated string quartet with period correction and changing tempo (Fig. 9). The initial tempo was 500 ms. The noise parameters were identical to the noise parameters of Fig. 6, but a few non-zero αs and βs were also included. Each

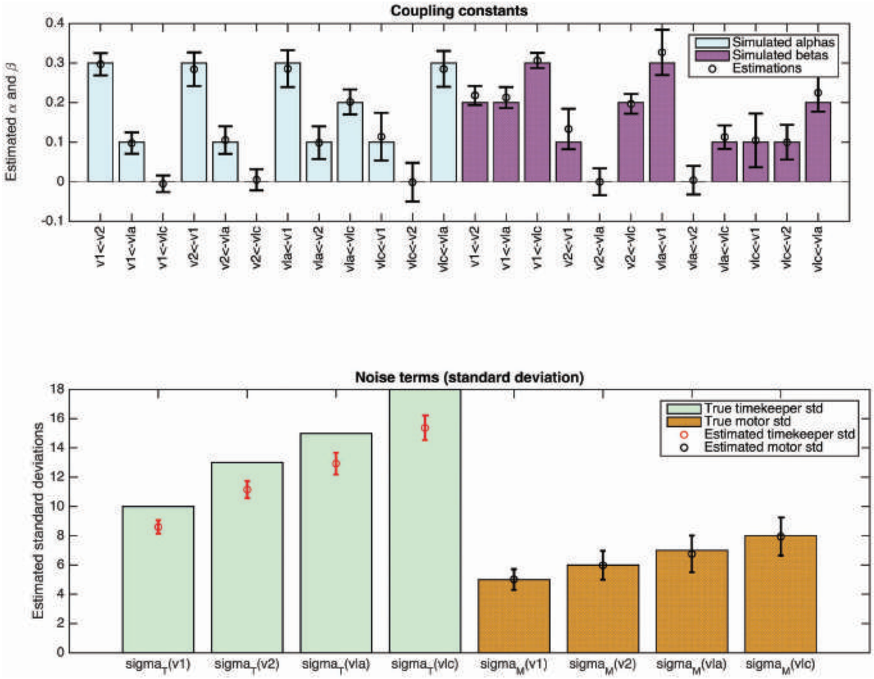

Figure 9. Simulated string quartet with period and phase correction. The top graphs show the phase and period constants of all four simulated players. The bottom graphs show the different timekeeper standard deviations (σ_T =10, 13, 15, and 18 for violin 1, violin 2, viola, and cello, respectively) and motor variances (σ_M =5, 6, 7, 8). Estimation of 1000 iterations of simulations with *nseq* = 16, and N = 40 are depicted in circles; error bars show the standard deviation of the estimates. Note the small bias in the estimation of the timekeeper, probably due a lack of sufficient data points in each of the simulated datasets. This figure is published in color in the online version.

simulated dataset had *nseq* = 16 repetitions of N = 40 beats similar to the setting used by Wing et al. (2014).

5.3. bGLS Method for Metrical Sequences

Previous work has shown that synchronization may be dependent on the hierarchical metric organization (Keller & Repp, 2005; Snyder et al., 2006; Vorberg & Hambuch, 1984). Metric organization refers to the hierarchical structuring of beats (salient regular pulsations), subdivisions of the beat, and groupings of beats into measures (London, 2012). These different levels of pulsations are typically nested in simple n:1 integer ratios such as 2:1 (duple meter), 3:1 (triple), or 4:1 (quadruple). Events at different metric positions within the hierarchy are perceived to differ in terms of accentuation (salience). In the case of a metric stimulus, it would be reasonable to consider a model that has different coupling constants (α and β) for different metric positions.

Formally, for the phase correction case this can be written[12]:

$$A_{k+1} + S_{k+1} = \left(1 - \alpha_{j_k}\right)A_k + T_k + M_{k+1} - M_k \tag{48}$$

where j_k is the index of the current metric position. In the case of quarter notes in a 4/4 meter (i.e., a meter with four quarter notes per measure), j_k is the sequence 1, 2, 3, 4, 1, 2, 3, 4, 1, ... Note that in this case we have four αs, one for each metric position.

More directly, this implies that:

$$\begin{aligned}
A_2 + S_2 &= (1 - \alpha_1)A_1 + T_1 + M_2 - M_1 \\
A_3 + S_3 &= (1 - \alpha_2)A_2 + T_2 + M_3 - M_2 \\
A_4 + S_4 &= (1 - \alpha_3)A_3 + T_3 + M_4 - M_3 \\
A_5 + S_5 &= (1 - \alpha_4)A_4 + T_4 + M_5 - M_4 \\
A_6 + S_6 &= (1 - \alpha_1)A_5 + T_5 + M_6 - M_5 \\
&\vdots \\
A_9 + S_9 &= (1 - \alpha_4)A_8 + T_8 + M_9 - M_8 \\
A_{10} + S_{10} &= (1 - \alpha_1)A_9 + T_9 + M_{10} - M_9 \\
&\vdots
\end{aligned} \tag{49}$$

Note that this model can also be written in a matrix notation as:

$$y = Bx + Z \tag{50}$$

where y and Z are exactly as in Sect. 1.3. But x is now a column vector of all possible αs:

$$x = \begin{bmatrix} 1 - \alpha_1 \\ 1 - \alpha_2 \\ 1 - \alpha_3 \\ 1 - \alpha_4 \end{bmatrix} \tag{51}$$

where α_i are the αs associated with each of the four possible metric positions.

[12] Similar formulae could be derived for period correction, but these are omitted here.

In this case B will have the following form:

$$B=\begin{bmatrix} A'_1 & 0 & 0 & 0 \\ 0 & A'_2 & 0 & 0 \\ 0 & 0 & A'_3 & 0 \\ 0 & 0 & 0 & A'_4 \\ A'_5 & 0 & 0 & 0 \\ 0 & A'_6 & 0 & 0 \\ 0 & 0 & A'_7 & 0 \\ 0 & 0 & 0 & A'_8 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{52}$$

where $A'_k = A_k - E(A)$.

Direct comparison shows that eqn. (48) and eqn. (50) are equivalent. We can now simply reapply the bGLS method in conjunction with variables *y, B,* and *x* as above.

This approach can also be used to analyze perturbation experiments with large unpredictable phase shifts (as may occur when a musician makes a rhythmic error). In this case, it may be useful to estimate the phase correction at different positions relative to the perturbation separately; for example, just above or later compared with the perturbation. We can write similar equations and again use the bGLS. This analysis was reported in Repp et al. (2012), where it was shown that phase correction just after the perturbation has a significantly larger phase correction than the phase correction at later stages.

6. Conclusion and Practical Suggestions for Applying Parameter Estimation to Real Data

In the current article and its companion paper (Jacoby et al., 2015), we described an algorithm for solving the parameter estimation problem for a canonical set of linear models of sensorimotor synchronization (Schulze et al., 2005; Vorberg & Schulze, 2002; Vorberg & Wing, 1996). In Jacoby et al. (2015), a limitation of the original estimation procedure was identified, and it was shown how to resolve this problem with a further assumption about the relationship between the timekeeper and the motor noise. In the current article, we went on to describe how to extend this algorithm to ensemble synchronization with and without period correction. Similar ideas could be extended to other models such as those suggested by Hary and Moore (1987a, b), Mates (1994a, b), Michon (1967), Pressing (1998a, b), Pressing and Jolley-Rogers (1997), and Thaut et al. (1998). Note that different models require re-parameterization and, often, different correlation structures (the covariance of the noise terms in the model).

Conceptually, the parameters of all of these models can be estimated with the bGLS method using ideas similar to those presented here[13].

Since the main point of the current paper was to show the mathematical validity of our proposed method, we relied largely on simulated datasets. However the bGLS method has already been applied on a dataset in two previous studies of sensorimotor synchronization (Jacoby & Repp, 2012; Repp, 2011).

We conclude with some practical suggestions intended to guide the choice of the correct model and data preparation based on experience in working with the method:

1. If there is no metronome (i.e., timing is self-paced), use Wing and Kristofferson (1973) and compute the motor and timekeeper variance from the inter-response interval *acvf*.
2. For single subject synchronizing with a metronome that is isochronous or contains occasional perturbations but no drift, use bGLS with phase correction (Sect. 2.3).
3. For the multi-person case with fixed tempo as in 2, use the variant in Sect. 3. Namely, use the bGLS algorithm but use instead of the variables y and B the one defined in Sect. 3.
4. For tempo changing sequences, use the variant of Sect. 4.1 for single subject and Sect. 5.1 for ensemble synchronization.
5. In all cases, compute the mean asynchrony based on maximal available data points (and of course, separately for each subject), since small errors in the mean asynchrony can cause additional biases[14].
6. In all cases, replace large deviating asynchronies (~200 ms at ISI of 500 ms) with the mean asynchrony [$E(A)$].
7. When multiple trials are available, estimate separately the parameters for each trial and take averages. Avoid using fewer than 20–30 time points (taps) per trial. Accurate estimates of parameters usually require at least 100 taps.
8. Always confirm that you have enough data and that you can extract the parameters properly by running validating simulations as done in this paper.
9. Estimates of motor variance that are small or zero occur in real datasets. To some extent, this can be expected from the variability of the estimates because the algorithm cannot return estimates less than zero. Preliminary

[13] See Appendix for an additional variant of the method which is required for developing bGLS estimates of some of these models (e.g., the one presented in Hary & Moore, 1987a,b).

[14] In all the datasets analysed in this paper, there were enough data to compute the mean asynchrony from one block. However, when working with the method for other datasets, we found scenarios where one block of data was not sufficient to estimate the mean asynchrony, and a significant improvement (in terms of bias and estimation error) was obtained when estimates of mean asynchronies are made based on the maximum available (and relevant) data points.

experiments (see Jacoby & Repp, 2012) show that in the case of period correction, zero motor variance is observed more often than expected from the model, which might suggest that some modifications are needed in the case of period correction. A detailed analysis of this phenomenon is beyond the scope of the current work. However, even if motor variance is slightly underestimated, the resulting effect and bias on the error correction parameters should be relatively small (as demonstrated in section 2.4 of this paper).

10. The minimal expected estimation error of the phase and period correction parameters in a typical sensorimotor synchronization experiment (sample size ~100 beats) is *above* $\pm$ 0.1. (This applies to all existing methods; it is not specific to our newly proposed method.) Specifying the parameters with higher accuracy should not be expected. However, note that this is a rough 'rule of thumb', which should be validated by running simulations that match the structure of the experimental data for which parameter estimates are sought.

We hope that these elaborate techniques as well as the MATLAB code of the estimation procedures, which can be found at *http://dx.doi.org/10.6084/m9.figshare.1391910*, will assist the community in using linear sensorimotor synchronization models for future data analysis.

The methods described in the current article can potentially be applied in several contexts. An obvious application is the study of the psychological processes underpinning interpersonal coordination in musical contexts. However, our methods could also be used to study individual differences in basic sensorimotor synchronization skills, which would have clinical implications for the assessment and treatment of patients with disorders affecting movement timing (e.g., Parkinson's disease and stroke). In both of the above applications, the efficiency of the methods described would be advantageous, as they enable real-time parameter estimation, which could potentially deliver online feedback to participants about their performance. More generally, the use of computational modeling to investigate sensorimotor synchronization skills is promising in that it provides a means to examine the psychological mechanisms that underlie this ubiquitous but not yet fully understood behavior.

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Appendix

A.1. An Alternative Formula for the Period Correction Model of Schulze et al. (2005)

Let us now examine an alternative bGLS algorithm for period correction. This algorithm is more complex and performs similarly to the one provided in the main paper. The advantage of this variant is that it can be used to extract parameters of models from the literature (e.g., Hary & Moore, 1987a,b) that could not be analyzed with the method provided in the paper.

Recall that the model of eqns (26) and (27) can be written as:

$$R_{k+1} = -\alpha A_k + t_k + H_k \tag{53}$$

where:

$$t_k = t_{k-1} - \beta A_k \tag{54}$$

$$H_k = T'_k + M_{k+1} - M_k \tag{55}$$

and $T'_k \sim N\left(0, \sigma_T^2\right), M_k \sim N\left(0, \sigma_M^2\right)$.

Now since this also implies that:

$$R_k = -\alpha A_{k-1} + t_{k-1} + H_{k-1} \tag{56}$$

then:

$$t_{k-1} = R_k + \alpha A_{k-1} - H_{k-1} \tag{57}$$

We can now substitute this formula into eqn. (54), obtaining:

$$t_k = R_k + \alpha A_{k-1} - H_{k-1} - \beta A_k \tag{58}$$

And this can be substituted into eqn. (53):

$$R_{k+1} = R_k - (\alpha + \beta)A_k + \alpha A_{k-1} + H_k - H_{k-1} \tag{59}$$

We denote by Z_k the following term:

$$Z_k = H_k - H_{k-1} = T_k^{'} - T_{k-1}^{'} + \left(M_{k+1} - 2M_k + M_{k-1}\right) \tag{60}$$

The period correction model therefore can be written as:

$$R_{k+1} = R_k - (\alpha + \beta)A_k + \alpha A_k + Z_k \tag{61}$$

A.2. *Comparing the Linear Period Correction Model and a Linear Second-Order Phase-Correction Model* (*Vorberg & Schulze, 2002*)

In their work related to the phase correction, Vorberg and Schulze (2002) also proposed a second-order phase correction model. Instead of a single phase correction constant dependent solely on previous asynchrony, they proposed a model that depends on the two previous asynchronies (which may be particularly relevant at fast tempi):

$$A_{k+1} = (1 - \alpha)A_k + \beta A_{k-1} + H_k \tag{62}$$

where: $H_k = T_k + M_{k+1} - M_k$, and α and β are the two phase correction constants.

However, using eqn. (61) it is easy to show that the period correction model in Schulze et al. (2005) that was analyzed in the previous subsection can be written as:

$$A_{k+1} = (2 - \alpha - \beta)A_k + (\alpha - 1)A_{k-1} + \left(S_k - S_{k-1}\right) + Z_k \tag{63}$$

where:

$$Z_k = H_k - H_{k-1} = T_k^{'} - T_{k-1}^{'} + (M_{k+1} - 2M_k + M_{k-1}) \tag{64}$$

Note the striking similarity between this model (eqn. 63) and the second-order phase correction model. In the case of an isochronous sequence $S(k)=S(k-1)$ the difference is a re-parameterization of the linear terms ($2-\alpha-\beta \rightarrow 1-\alpha; \alpha-1 \rightarrow \beta$) and a different structure of the noise term (Z_k instead of H_k).

A.3. Writing the Period Correction Model in Matrix Notation and Applying the bGLS Estimation Method

The auto-covariance of Z_k has the following form:

$$\begin{aligned}
\gamma_z(0) &= \sigma_Z^2 = 2\sigma_T^2 + 6\sigma_M^2 \\
\gamma_z(1) &= -\sigma_T^2 - 4\sigma_M^2 \\
\gamma_z(2) &= \sigma_M^2
\end{aligned}$$

and $\gamma_z(k)=0$ for $k > 2$.

With this formalism, eqn. (61) can be written in matrix notation:

$$y = \begin{bmatrix} R_3 - R_2 \\ \vdots \\ R_{N+1} - R_N \end{bmatrix} = \begin{bmatrix} A_2 - E(A) & A_1 - E(A) \\ \vdots & \vdots \\ A_N - E(A) & A_{N-1} - E(A) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} Z_2 \\ \vdots \\ Z_N \end{bmatrix} = Bx + Z \tag{65}$$

where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -\alpha - \beta \\ \alpha \end{bmatrix}$, and $Z \sim N(\mu, \Sigma)$ multivariate Gaussian variable with zero mean ($\mu = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$) and has a Toeplitz covariance matrix of the following form:

$$\Sigma_Z = \begin{pmatrix}
2\sigma_T^2 + 6\sigma_M^2 & -\sigma_T^2 - 4\sigma_M^2 & \sigma_M^2 & 0 & \cdots & 0 \\
-\sigma_T^2 - 4\sigma_M^2 & \ddots & \ddots & \ddots & & \vdots \\
\sigma_M^2 & \ddots & \ddots & \ddots & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \sigma_M^2 \\
\vdots & & \ddots & \ddots & \ddots & -\sigma_T^2 - 4\sigma_M^2 \\
0 & \cdots & 0 & \sigma_M^2 & -\sigma_T^2 - 4\sigma_M^2 & 2\sigma_T^2 + 6\sigma_M^2
\end{pmatrix} \tag{66}$$

Note that there are few constraints on this matrix: we know that $\sigma_M^2 > 0$ and $\sigma_T^2 > 0$ and we further assume that $\sigma_T > \sigma_M$ (if this assumption it not made, we will suffer from the Cramér–Rao lower bound, as explained in Jacoby et al., 2015, since the problem of parameter interdependence grows with increasing degrees of freedom).

Since $\sigma_M^2 = \frac{\gamma_z(0) + 2\gamma_z(1)}{-2}$ and $\sigma_T^2 = 2\gamma_z(0) + 3\gamma_z(1)$

we know that: $\frac{\gamma_z(0) + 2\gamma_z(1)}{-2} > 0$ or $\gamma_z(1) < -\frac{\gamma_z(0)}{2}$

and that: $2\gamma_z(0) + 3\gamma_z(1) > 0$ or $\gamma_z(1) > -\frac{2}{3}\gamma_z(0)$

and that: $2\gamma_z(0) + 3\gamma_z(1) > \frac{\gamma_z(0) + 2\gamma_z(1)}{-2}$ or $\gamma_z(1) > -\frac{5}{8}\gamma_z(0)$.

Note that if we assume that: $\sigma_T > L \cdot \sigma_M$ (so far we assumed $L = 1$) then:

$$\gamma_z(1) > -\frac{(4 + L)}{(2L + 6)}\gamma_z(0) \tag{67}$$

To conclude, in order to satisfy all three conditions it is sufficient to ensure that:

$$-\frac{\gamma_z(0)}{2} > \gamma_z(1) > -\frac{5}{8}\gamma_z(0) \text{ and } \gamma_z(2) = \sigma_M^2 = \frac{\gamma_z(0) + 2\gamma_z(1)}{-2} \tag{68}$$

This leads to the following method.

A Variant of the bGLS Method for Period Correction

- Input: *nseq* sequences of N taps with asynchronies $\{A_{k=1 \ldots N}\}_{i=1 \ldots nseq}$ and inter-response intervals $\{R_{k=1 \ldots N+1}\}_{i=1 \ldots nseq}$.
- *niter*, a constant that determines the number of iterations.

Output: parameter estimates $\left(\overline{\alpha}, \overline{\beta}, \overline{\sigma}_T, \overline{\sigma}_M\right)$

1. Compute the grand average $E(A)$ from all the data.
2. For each sequence of the *nseq* sequences compute:

$$y = \begin{bmatrix} R_3 - R_2 \\ \vdots \\ R_{N+1} - R_N \end{bmatrix}$$

$$B = \begin{bmatrix} A_2 - E(A) & A_1 - E(A) \\ \vdots & \vdots \\ A_N - E(A) & A_{N-1} - E(A) \end{bmatrix}$$

2.2. Start by setting $\Sigma_1 = 2 \cdot I + (-1) \cdot \Delta$, where I is the $(N-1)$-by-$(N-1)$ identity matrix and Δ is a $(N-1)$-by-$(N-1)$ matrix with one on the two secondary diagonals and 0 elsewhere.

2.3. Iterate the following equations:

(i) Set: $x_n = \left(B^T \Sigma_n^{-1} B\right)^{-1} \left(B^T \Sigma_n^{-1}\right) y$

(ii) Set: $d_n = y - Bx_n$

(iii) Set: $\gamma_n(0) = \frac{1}{N} \Sigma_{k=1}^{N} d_n(k) d_n(k)$

(iv) Set: $\gamma_n(1) = \frac{1}{N-1} \Sigma_{k=1}^{N-1} d_n(k+1) d_n(k)$

(v) Adjust $\gamma_n(1)$ by increasing or decreasing it so that

$$-\frac{\gamma_n(0)}{2} > \gamma_n(1) > -\frac{5}{8}\gamma_n(0)$$

(vi) Set: $\gamma_n(2) = \frac{\gamma_n(0) + 2\gamma_n(1)}{-2}$

(vii) Set: $\Sigma_{n+1} = \gamma_n(0) I + \gamma_n(1) \Delta_1 + \gamma_n(2) \Delta_2$

Δ_1 and Δ_2 are $(N-1)$-by-$(N-1)$ matrices with one on the two secondary and tertiary diagonals and 0 elsewhere, respectively.

a. Set: $\overline{\alpha} = x_{2,niter}$, $\overline{\beta} = -\left(x_{1,niter} + x_{2,niter}\right)$ $\overline{\sigma}_M = \sqrt{\frac{\gamma_{niter}(0) + 2\gamma_{niter}(1)}{-2}}$ and $\overline{\sigma}_T = \sqrt{2\gamma_{niter}(0) + 3\gamma_{niter}(1)}$

3. Take the average values obtained from the *nseq* sequences.

Numerical simulations show that this algorithm performs comparably to the method provided in the main body of the paper.